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THE DIRICHLET PROBLEM FOR NONLOCAL ELLIPTIC OPERATORS WITH $C^{0,\alpha}$ EXTERIOR DATA

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Abstract. In this note we study the boundary regularity of solutions to nonlocal Dirichlet problems of the form $Lu = 0$ in Ω, $u = g$ in $\mathbb{R}^N \setminus \Omega$, in nonsmooth domains Ω. When g is smooth enough, then it is easy to transform this problem into an homogeneous Dirichlet problem with a bounded right-hand side for which the boundary regularity is well understood. Here, we study the case in which $g \in C^{0,\alpha}$, and establish the optimal Hölder regularity of u up to the boundary. Our results extend previous results of Grubb for C^∞ domains Ω.

1. Introduction

Given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^N$, we study the regularity of solutions to nonlocal Dirichlet problems of the form

$$\begin{cases} Lu = 0 & \text{in } \Omega, \\ u = g & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

where L is an operator of the form

$$-Lu(x) = \int_{\mathbb{R}^N} \left(u(x) - \frac{u(x+y) + u(x-y)}{2} \right) K(y)dy, \tag{1.2}$$

with kernel K satisfying

$$K(y) = K(-y) \quad \text{and} \quad \frac{\lambda}{|y|^{N+2s}} \le K(y) \le \frac{\Lambda}{|y|^{N+2s}}, \quad y \in \mathbb{R}^N \setminus \{0\}. \tag{1.3}$$

Here, $s \in (0,1)$ and $0 < \lambda \le \Lambda$.

In most of our results we will assume in addition that

$$K \text{ is homogeneous.} \tag{1.4}$$

Notice that, when $\lambda = \Lambda$, we recover (a multiple of) the fractional Laplacian $(-\Delta)^s$. Even in that case, the results that we establish in this paper were only known for C^∞ domains Ω; see Grubb [11, Theorem 2.5].

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The existence, regularity, and further properties of solutions to nonlocal Dirichlet problems of this type have been an active topic of research in the last years; we refer to [1, 3, 4, 6, 9, 14, 15] and references therein.

When g is smooth enough (e.g., $g \in C^{2s+\varepsilon}$ for some $\varepsilon > 0$) then it is easy to transform (1.1) into a homogeneous Dirichlet problem of the type $Lu = f$ in Ω, $u = 0$ in Ω^c, with $f \in L^\infty(\Omega)$. It is well known then that (as long as Ω is smooth enough) solutions u are $C^{0,s}$ up to the boundary.

However, when g is less regular (e.g., $g \in C^{0,\alpha}$) then the boundary regularity of solutions to (1.1) has to be treated more carefully, and to the best of our knowledge this has only been studied in the case of C^∞ domains by [11].

Roughly speaking, the aim of this paper is to show that, when $g \in C^{0,\alpha}$ ($\alpha > 0$ small) and Ω is Lipschitz, then u is $C^{0,\alpha}$ up to the boundary. This is explained in more detail next.

1.1. **The local case.** When $s = 1$, solutions to the Dirichlet problem

$$\begin{cases} \Delta u &= 0 \quad \text{in } \Omega, \\ u &= g \quad \text{on } \partial\Omega \end{cases}$$

satisfy the following (cf. [10, 12]):

(a) If $g \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha \in (0, 1)$, and Ω is at least C^1, then $u \in C^{0,\alpha}(\overline{\Omega})$.
(b) If $g \in C^{0,1}(\partial\Omega)$, then in general $u \notin C^{0,1}(\overline{\Omega})$, even if Ω is of class C^∞.
(c) If $g \in C^{1,\alpha}(\partial\Omega)$ for some $\alpha \in (0, 1)$, and Ω is at least $C^{1,\alpha}$, then $u \in C^{1,\alpha}(\overline{\Omega})$.

Finally, when Ω is not C^1 but only Lipschitz, we have the following:

(d) If Ω is Lipschitz, then there exists $\alpha_0 = \alpha_0(\Omega) > 0$ such that if $g \in C^{0,\alpha}(\partial\Omega)$ for some $\alpha \in (0, \alpha_0]$, then $u \in C^{0,\alpha}(\overline{\Omega})$.

The above results are sharp in terms of the regularity of g, and also in terms of the regularity of Ω.

1.2. **Our results.** The goal of this paper is to provide analogous results to (a), (b), (c), and (d) for nonlocal Dirichlet problems of the type (1.1), with $s \in (0, 1)$.

The right assumption on the exterior datum g turns out to be

$$|g(x) - g(z)| \leq C_0|x - z|^\alpha \quad \text{for all } x \in \mathbb{R}^N \setminus \Omega,\ z \in \partial\Omega, \tag{1.5}$$

for some constant C_0 and $\alpha \in (0, 1)$. Notice that, in particular, g is $C^{0,\alpha}$ on $\partial\Omega$ (but not necessarily outside $\overline{\Omega}$). Moreover, taking C_0 larger if necessary, g will satisfy the growth condition

$$|g(x)| \leq C_0(1 + |x|^\alpha), \qquad x \in \mathbb{R}^N \setminus \Omega. \tag{1.6}$$

Our first (and main) result provides the analogue of property (a) above.

Theorem 1.1 ($\alpha < s$). *Let $\Omega \subset \mathbb{R}^N$ be any bounded C^1 domain, $s \in (0, 1)$, L as in (1.2)–(1.3)–(1.4), and g as in (1.5)–(1.6), with $\alpha \in (0, s)$. Then the solution u to (1.1) is of class $C^{0,\alpha}(\overline{\Omega})$, with*

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} \leq CC_0,$$

where C depends only on n, s, λ, Λ, α, and Ω.

Moreover, we will show that the previous result fails when $\alpha = s$, even if Ω is smooth. This is the analogue of property (b) above.

Proposition 1.2 ($\alpha = s$)**.** *Let $s \in (0,1)$ and $-L = (-\Delta)^s$. Then, there exist a C^∞ domain $\Omega \subset \mathbb{R}^2$ and a function g satisfying* (1.5)–(1.6) *with $\alpha = s$, such that the solution u to* (1.1) *satisfies $u \notin C^{0,s}(\overline{\Omega})$.*

When $\alpha > s$, using known results from [18], we will establish the following. Notice that, for nonlocal operators of this type, the best Hölder regularity one can get is $C^{0,s}(\overline{\Omega})$, even if g and Ω are C^∞; see [15]. This is why the analogue of property (c) above reads as follows.

Proposition 1.3 ($\alpha > s$)**.** *Let $\Omega \subset \mathbb{R}^N$ be any bounded $C^{1,\gamma}$ domain, $\gamma > 0$, $s \in (0,1)$, L as in* (1.2)–(1.3)–(1.4)*, and g as in* (1.5)–(1.6)*, with $\alpha > s$ and $\alpha < 2s$. Then the solution u to* (1.1) *is of class $C^{0,s}(\overline{\Omega})$, with*

$$\|u\|_{C^{0,s}(\overline{\Omega})} \leq CC_0,$$

where C depends only on n, s, λ, Λ, α, and Ω.

Finally, when Ω is of class $C^{0,1}$ we establish the following result, analogue to (d) above. Notice that here we do not need to assume that the kernel K is homogeneous.

Theorem 1.4 ($\partial\Omega \in \mathrm{Lip}$)**.** *Let $\Omega \subset \mathbb{R}^N$ be any bounded Lipschitz domain, $s \in (0,1)$, and L as in* (1.2)–(1.3)*. Then, there exists $\beta_0 > 0$, depending only on Ω, s, λ, and Λ, such that the following holds. Let g be as in* (1.5)–(1.6)*, with $\alpha \in (0, \beta_0]$. Then the solution u to* (1.1) *is of class $C^{0,\alpha}(\overline{\Omega})$, with*

$$\|u\|_{C^{0,\alpha}(\overline{\Omega})} \leq CC_0,$$

where C depends only on n, s, λ, Λ, α, and Ω.

The strategy in our proof of the $C^{0,\alpha}$ regularity of u is as follows. The basic idea is to extend the exterior data g to a function $\overline{g}$, defined in $\mathbb{R}^N$, and such that it is as regular as it can be inside Ω. Then, we show that $|L\overline{g}| \leq Cd^{\alpha-2s}$ in Ω, where $d(x) := \mathrm{dist}\big(x, \mathbb{R}^N \setminus \Omega\big)$. Thanks to this, defining $v = u - \overline{g}$, we are led to the study of the problem

$$\left\{ \begin{array}{rcll} Lv & = & f & \text{in } \Omega, \\ v & = & 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{array} \right. \tag{1.7}$$

with $|f| \leq Cd^{\alpha-2s}$ in Ω. We then prove regularity properties up to the boundary of solutions v to (1.7), and show that

$$\|v\|_{C^{0,\alpha}(\overline{\Omega})} \leq C\|d^{2s-\alpha} f\|_{L^\infty(\Omega)}$$

for some $C > 0$. To do this, we need to construct fine barriers, which must take into account two important features: first, f is very singular near the boundary $\partial\Omega$; and second, the domain Ω is only C^1 (or $C^{0,1}$).

Remark 1.5. In the statements above, we have not specified the nature of the solutions we consider (weak or viscosity). Whenever weak or viscosity solutions exist, our results apply to them. Actually, using our new regularity results it is easy to prove the existence of a viscosity solution to the problem (1.1) under our assumptions on L, Ω, and g; see Corollary 4.2.

The paper is organized as follows. In Section 2 we present some preliminary results we will employ later in the main proofs. Section 3 is the core of the paper: we prove Theorem 1.1 and Propositions 1.2 and 1.3. Finally, in Section 4 we consider domains of class $C^{0,1}$, showing Theorem 1.4.

2. Preliminary results

This section is devoted to the proof of some preliminary results. The first one is an L^∞ bound on (weak) solutions, based on the maximum principle.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^N$ be any bounded domain, $s \in (0,1)$, $\alpha \in (0,2s)$, L as in* (1.2)–(1.3)*, and g satisfying* (1.6)*. Then, the solution u to* (1.1) *is bounded in Ω and satisfies*

$$\|u\|_{L^\infty(\Omega)} \le CC_0,$$

with C depending only on N, s, α, Ω, and the ellipticity constants.

Proof. First notice that, dividing by a constant if necessary, we may assume that $C_0 = 1$. Since the function g is not assumed to be bounded, the boundedness of u in Ω does not follow immediately. Thus, we construct an equivalent version of problem (1.1), with a nontrivial r.h.s. $f \in L^\infty(\Omega)$ and a new exterior data $g_1 \in L^\infty(\mathbb{R}^N \setminus \Omega)$. Then we will apply some known L^∞ estimates, obtaining the proof of our statement.

Let $R_0 > 0$ be such that $\Omega \subset B_{R_0/2}$, and we consider the functions

$$g_1(x) := g\,\chi_{B_{R_0}\setminus\Omega} \qquad \text{and} \qquad g_2 := g\,\chi_{\mathbb{R}^N\setminus B_{R_0}},$$

so that $g = (g_1 + g_2)|_{\mathbb{R}^N\setminus\Omega}$. The function $w := u - g_2$ satisfies

$$\begin{cases} -Lw = f & \text{in } \Omega, \\ \quad w = g_1 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases}$$

with $f := Lg_2$. Now, since g_2 is supported outside B_{R_0}, and $B_{R_0} \supset\supset \Omega$, then it is easy to check that $|f| \le C$ in Ω. Therefore, since $|g_1| \le C$ in $\mathbb{R}^N \setminus \Omega$, it follows from [15, Corollary 5.2] that $w \in L^\infty(\Omega)$, and thus $u \in L^\infty(\Omega)$. $\square$

The second gives an extension g in Ω, which is as smooth as possible inside Ω.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^N$ be any bounded C^1 domain, and let g be as in* (1.5)*, with $\alpha \in (0,1)$. Then, there exists a function $\overline{g} \in C^{0,\alpha}(\overline{\Omega}) \cap C^\infty(\Omega)$ such that*

$$\begin{aligned} \overline{g} &= g && \text{in } \mathbb{R}^N \setminus \Omega, \\ |D^2\overline{g}| &\le Cd^{\alpha-2} && \text{in } \Omega, \end{aligned} \tag{2.1}$$

where C depends only on N, α, and Ω.

Proof. We consider the solution of $\Delta\overline{g} = 0$ in Ω, $\overline{g} = g$ on $\partial\Omega$. Since $g \in C^{0,\alpha}(\partial\Omega)$, and Ω is of class C^1, it follows from standard regularity theory that $\overline{g} \in C^\infty(\Omega) \cap C^{0,\alpha}(\overline{\Omega})$ and that[1] $|D^2\overline{g}| \le Cd^{\alpha-2}$ in Ω. $\square$

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^N$ be any bounded Lipschitz domain. Then, there exists $\alpha_0 = \alpha_0(\Omega)$ such that the following holds. Let g be as in* (1.5)*, with $\alpha \in (0,1)$. Then, there exists a function $\overline{g} \in C^{0,\beta}(\overline{\Omega}) \cap C^\infty(\Omega)$ such that*

$$\begin{aligned} \overline{g} &= g && \text{in } \mathbb{R}^N \setminus \Omega, \\ |D^2\overline{g}| &\le Cd^{\beta-2} && \text{in } \Omega, \end{aligned} \tag{2.2}$$

with $\beta := \min\{\alpha, \alpha_0\}$. The constant $C > 0$ depends only on N, α, and Ω.

Proof. The proof is that of Lemma 2.2, recalling that when Ω is Lipschitz, then the harmonic extension of $g \in C^{0,\alpha}(\partial\Omega)$ satisfies $\overline{g} \in C^{0,\beta}(\overline{\Omega})$, with $\beta = \min\{\alpha, \alpha_0\}$. $\square$

[1]This can be shown by using that $\overline{g} \in C^{0,\alpha}(\overline{\Omega})$, standard elliptic regularity estimates, and the fact that the function $\overline{g} - \overline{g}(x_0)$ is harmonic in Ω for any choice of x_0.

We next compute the operator L evaluated on the extension $\overline{g}$ constructed above.

Lemma 2.4. (a) *Let $\Omega \subset \mathbb{R}^N$ be any bounded C^1 domain, $s \in (0,1)$, $\alpha \in (0,2s)$, L as in* (1.2)*–*(1.3)*–*(1.4)*, g as in* (1.5)*–*(1.6)*, and $\overline{g}$ given by Lemma* 2.2*. Then,*

$$|L\overline{g}| \le CC_0 d^{\alpha-2s} \quad \text{in } \Omega$$

for some constant C depending only on N, s, α, Ω, and the ellipticity constants.

(b) *Let $\Omega \subset \mathbb{R}^N$ be any bounded Lipschitz domain, $s \in (0,1)$, $\alpha \in (0,2s)$, L as in* (1.2)*–*(1.3)*, g as in* (1.5)*–*(1.6)*, and $\overline{g}$ and α_0 given by Lemma* 2.3*. Then,*

$$|L\overline{g}| \le CC_0 d^{\beta-2s} \quad \text{in } \Omega,$$

where $\beta = \min\{\alpha, \alpha_0\}$. The constant C depends only on N, s, α, Ω, and the ellipticity constants.

Proof. We prove (a)—the same proof works for (b) replacing α by β. As before, we may assume $C_0 = 1$. Let $x_0 \in \Omega$, and define $\varrho := d(x_0)$. Notice that we may assume $\varrho \in (0, \varrho_0)$ for some small $\varrho_0 > 0$—since $\overline{g} \in C^\infty(\Omega)$, the result is obvious if $d(x_0) \ge \varrho_0 > 0$.

Now, up to a positive multiplicative constant, we write

$$\begin{aligned} L\overline{g}(x_0) = &\frac{1}{2}\int_{B_{\varrho/2}} (\overline{g}(x_0+y) + \overline{g}(x_0-y) - 2\overline{g}(x_0))K(y)dy \\ &+ \frac{1}{2}\int_{\mathbb{R}^N\setminus B_{\varrho/2}} (\overline{g}(x_0+y) + \overline{g}(x_0-y) - 2\overline{g}(x_0))K(y)dy =: \frac{1}{2}I_1 + \frac{1}{2}I_2. \end{aligned}$$

We notice that when $\alpha > s$, it is crucial to have also $\alpha < 2s$ so that the second integral above is finite. Up to taking $\varrho_0 > 0$ smaller, the first integral can be estimated by using (2.1) as follows:

$$|I_1| \le C\int_{B_{\varrho/2}} \frac{|D^2\overline{g}(x_0)||y|^2}{|y|^{N+2s}}dy \le C\varrho^{\alpha-2}\int_{B_{\varrho/2}} |y|^{2-N-2s}dy = C\varrho^{\alpha-2s}.$$

To estimate the second integral, we pick a point $z_0 \in \partial\Omega$ such that $|x_0 - z_0| = \varrho$, and we consider

$$\begin{aligned} |I_2| &\le C\int_{\mathbb{R}^N\setminus B_{\varrho/2}} \frac{|\overline{g}(x_0+y) - \overline{g}(z_0)| + |\overline{g}(x_0-y) - \overline{g}(z_0)| + 2|\overline{g}(x_0) - \overline{g}(z_0)|}{|y|^{N+2s}}dy \\ &\le C\int_{\mathbb{R}^N\setminus B_{\varrho/2}} \frac{|x_0 - z_0 + y|^\alpha + |x_0 - z_0 - y|^\alpha + 2|x_0 - z_0|^\alpha}{|y|^{N+2s}}dy \\ &\le C\int_{\mathbb{R}^N\setminus B_{\varrho/2}} \frac{(\varrho + |y|)^\alpha + \varrho^\alpha}{|y|^{N+2s}}dy + C\int_{\varrho/2}^\infty \frac{r^\alpha(\varrho/r+1)^\alpha + \varrho^\alpha}{r^{1+2s}}dr \\ &\le C\int_{\varrho/2}^\infty r^{\alpha-1-2s}dr + C\varrho^\alpha\int_{\varrho/2}^\infty r^{-1-2s}dr = C\varrho^{\alpha-2s}. \end{aligned}$$

In the second inequality we used that, since $\overline{g} \in C^{0,\alpha}(\overline{\Omega})$, if $x_0 \pm y \in \Omega$, then

$$|\overline{g}(x_0 \pm y) - \overline{g}(z_0)| \le C_0|x_0 - z_0 \pm y|^\alpha$$

while if $x_0 \pm y \in \mathbb{R}^N \setminus \Omega$, the same inequality holds by the assumption (1.5) on g. The third one follows since $|x_0 - z_0 \pm y| \le |x_0 - z_0| + |y| = \varrho + |y|$, while the last one since $\varrho/r \le 2$. Combining the estimates on I_1 and I_2, the lemma follows. $\square$

We end the section with the following lemma.

Lemma 2.5. *Let $s \in (0,1)$, $\alpha \in (0,s)$, $\nu \in \mathbb{S}^{N-1}$, and L be as in* (1.4). *Then the function $\varphi_\nu^\alpha(x) := (x\cdot\nu)_+^\alpha$ satisfies*

$$-L\varphi_\nu^\alpha(x) > 0 \quad \textit{in } \{x\cdot\nu > 0\}.$$

Proof. First, note that $\varphi_\nu(x) = u(x\cdot\nu)$, where $u(t) := (t_+)^\alpha$. Consequently (cf. [16, Lemma 2.1 and Lemma 2.3]), it follows that

$$-L\varphi_\nu(x) = \left(c_s\int_{\mathbb{S}^{N-1}} |\vartheta_n|^{2s}K(\theta)d\theta\right)(-\Delta)^s_{\mathbb{R}}u(x\cdot\nu) \quad \text{in } \{x\cdot\nu>0\},$$

where $c_s > 0$ is a suitable constant. Therefore, it suffices to prove the result for the fractional Laplacian $(-\Delta)^s_{\mathbb{R}}$ in dimension $N=1$.

For this, notice that u is homogeneous of degree α, and hence Lu is homogeneous of degree $\alpha-2s$. Thus, it is enough to show that

$$(-\Delta)^s_{\mathbb{R}}u(x_0) > 0 \quad \text{for some } x_0>0.$$

We consider the function $v_c(x) := (x+c)_+^s$, which satisfies $(-\Delta)^s_{\mathbb{R}}v_c(x) = 0$ for all $x > -c$, where $c>0$ is a free parameter (cf. [16, Lemma 2.2]). Since $v_c \ge u$ in $\mathbb{R}$ for c large enough, it easy to see that there are $x_0>0$ and $c>0$ such that

$$v_c(x_0) = u(x_0) \qquad \text{and} \qquad v_c \ge u \quad \text{in } \mathbb{R}.$$

Consequently, it follows that

$$v_c(x_0) - \frac{v_c(x_0+y)+v_c(x_0-y)}{2} \le u_c(x_0) - \frac{u_c(x_0+y)+u_c(x_0-y)}{2}, \quad y\in\mathbb{R}$$

(with strict inequality in $\mathbb{R}_+\setminus\{x_0\}$), and so $0 = (-\Delta)^s_{\mathbb{R}}v_c(x_0) < (-\Delta)^s_{\mathbb{R}}u(x_0)$. $\square$

3. Proof of the main results

The main goal of this section is to prove Theorem 1.1. For this, we will use the following.

Lemma 3.1 ([13])**.** *Let $\Omega\subset\mathbb{R}^N$ be any bounded C^1 domain. Then, there exists a modulus of continuity ω and a function $\psi\in C^1(\overline{\Omega})$ satisfying*

$$\begin{aligned} &C^{-1}d \le \psi \le Cd \quad \textit{in } \Omega,\\ &|\nabla\psi(x)-\nabla\psi(y)| \le \omega(|x-y|) \quad \textit{for all } x,y\in\Omega,\\ &|D^2\psi(x)| \le \omega(d(x))\,d^{-1}(x) \quad \textit{for all } x\in\Omega, \end{aligned} \tag{3.1}$$

where $d(x) = \mathrm{dist}(x,\Omega^c)$ and $C>0$ is a constant depending only on Ω.

In the case of $C^{1,\gamma}$ domains Ω, it is easy to see that one can choose $\omega(r) = Cr^\gamma$ in (3.1); see [18, Definition 2.1]. For a proof in the case of general C^1 domains, we refer to [13, Lemma 1.1 and Theorem 2.1].

We next prove two technical lemmas—in the case that Ω is $C^{1,\gamma}$, they correspond to Lemmas 2.4 and 2.5 of [18].

Lemma 3.2. *Let Ω be any C^1 domain, and let ψ and ω be defined as in* (3.1). *Then, for each $x_0\in\Omega$, it holds that*

$$\Big|\psi(x_0+y) - \big(\psi(x_0)+\nabla\psi(x_0)\cdot y\big)_+\Big| \le C\omega(|y|)\,|y|, \quad y\in\mathbb{R}^N,$$

where $C>0$ depends only on Ω.

Proof. Since $\psi \in C^1(\overline{\Omega})$, there is an extension $\widetilde{\psi} \in C^1(\mathbb{R}^N)$ with $\widetilde{\psi} \le 0$ in $\mathbb{R}^N \setminus \Omega$ and $\widetilde{\psi}|_\Omega = \psi$, preserving also the modulus of continuity ω of ψ (up to a multiplicative constant). Thus, if $x_0 \in \Omega$ we have

$$\begin{aligned}\left|\widetilde{\psi}(x) - \widetilde{\psi}(x_0) - \nabla\widetilde{\psi}(x_0)\cdot(x - x_0)\right| &= \left|\Big(\nabla\widetilde{\psi}(\lambda x + (1-\lambda)x_0) - \nabla\widetilde{\psi}(x_0)\Big)\cdot(x - x_0)\right| \\ &\le \left|\nabla\widetilde{\psi}(\lambda x + (1-\lambda)x_0) - \nabla\widetilde{\psi}(x_0)\right| |x - x_0| \\ &\le C\omega(\lambda|x - x_0|)|x - x_0| \le C\omega(|x - x_0|)|x - x_0|\end{aligned}$$

for all $x \in \mathbb{R}^N$, since $\lambda \in (0,1)$ and ω is increasing. Now, using that $\widetilde{\psi}(x_0) = \psi(x_0)$, $\nabla\widetilde{\psi}(x_0) = \nabla\psi(x_0)$, $(\widetilde{\psi})_+ = \psi$, and that $|a_+ - b_+| \le |a - b|$, we obtain

$$\left|\psi(x) - \big(\psi(x_0) + \nabla\psi(x_0)\cdot(x - x_0)\big)_+\right| \le \omega(|x - x_0|)|x - x_0|$$

for all $x \in \mathbb{R}^N$, and the thesis follows. □

The next lemma is similar to [18, Lemma 2.5].

Lemma 3.3. *Let Ω be any C^1 domain, $x_0 \in \Omega$, $\varrho = d(x_0)/2$, and ω a modulus of continuity. Then, there exists a modulus of continuity $\widetilde{\omega}$ such that, if $\delta > -1$ and $\beta \neq \delta$, then*

$$\int_{B_1\setminus B_{\varrho/2}} d^\delta(x_0 + y)\frac{\omega(|y|)dy}{|y|^{N+\beta}} \le C\left(1 + \widetilde{\omega}(\varrho)\varrho^{\delta-\beta}\right)$$

for some constant $C > 0$ depending only on δ, β, Ω, and ω.

Proof. Let us take $x_0 = 0$ (this can always be done up to a translation of the coordinate system), define $\varrho = d(0)/2$, and take $\kappa_* > 0$ such that the level sets $\{d = t\}$ are C^1 for all $t \in (0, \kappa_*]$ (this κ_* exists since $\Omega \in C^1$). Without loss of generality, we can assume $\kappa_* > 2\varrho$ (i.e., $\varrho > 0$ small). Notice that if $\varrho \ge \varrho_0 > 0$ the inequality in our statement is just

$$\int_{B_1\setminus B_{\varrho/2}} d^\delta(x_0 + y)\frac{\omega(|y|)dy}{|y|^{N+\beta}} \le C,$$

and it is immediately verified. So, from now on, we will assume $0 < \varrho < \varrho_0$ for some small ϱ_0. First of all, we have

$$\int_{(B_1\setminus B_{\varrho/2})\cap\{d\ge\kappa_*\}} d^\delta(y)\frac{\omega(|y|)dy}{|y|^{N+\beta}} \le C, \tag{3.2}$$

where $C > 0$ depends only on δ, β, Ω, and ω, thanks to the choice $\kappa_* > 2\varrho$.

Now we fix $M > M_0$ such that $2^{-M} \le \varrho \le 2^{-M+1}$ (M_0 is large and depends on $\varrho_0 > 0$) and, using the coarea formula, we obtain

$$\begin{aligned}\int_{(B_1\setminus B_{\varrho/2})\cap\{d<\kappa_*\}} d^\delta(y)\frac{\omega(|y|)}{|y|^{N+\beta}}dy &\le \sum_{k=0}^{M}\int_{(B_{2^{-k}}\setminus B_{2^{-k-1}})\cap\{d<\kappa_*\}} d^\delta(y)\frac{\omega(|y|)}{|y|^{N+\beta}}dy \\ &\le \sum_{k=0}^{M}\frac{\omega(2^{-k})}{2^{-(N+\beta)k}}\int_{(B_{2^{-k}}\setminus B_{2^{-k-1}})\cap\{d<c2^{-k}\}} d^\delta(y)|\nabla d(y)|dy \\ &= \sum_{k=0}^{M}\frac{\omega(2^{-k})}{2^{-(N+\beta)k}}\int_0^{c2^{-k}} t^\delta dt\int_{(B_{2^{-k}}\setminus B_{2^{-k-1}})\cap\{d=t\}} d\mathcal{H}^{N-1}(y)\end{aligned}$$

for some $c > 0$ depending only on Ω. Now, since $\Omega \in C^1$, we have for each $t \in (0, \kappa_*)$

$$\mathcal{H}^{N-1}\big((B_{2^{-k}} \setminus B_{2^{-k-1}}) \cap \{d = t\}\big) \le C 2^{-(N-1)k}$$

for some constant $C > 0$ depending only on Ω. Consequently, it follows that

$$\int_{(B_1 \setminus B_{\varrho/2}) \cap \{d < \kappa_*\}} d^\delta(y) \frac{\omega(|y|) dy}{|y|^{N+\beta}} \le C \sum_{k=0}^{M} \omega(2^{-k}) 2^{(\beta-\delta)k}.$$

Thus if $\beta < \delta$, it is immediate to see that the above sum is bounded independently of M (i.e., ϱ). Keeping in mind that $2^{-M} \le \varrho \le 2^{-M+1}$, when $\beta > \delta$, it is enough to prove the existence of a modulus of continuity $\widetilde{\omega}$ such that

$$\sum_{k=0}^{M} \omega(2^{-k}) 2^{(\beta-\delta)k} \le \widetilde{w}(2^{-M})\, 2^{(\beta-\delta)M}. \tag{3.3}$$

Finally, thanks to the Stolz-Cesaro theorem (l'Hopital rule for sequences), we obtain

$$\begin{aligned}
&\lim_{M\to\infty} \frac{\sum_{k=0}^{M} \omega(2^{-k}) 2^{(\beta-\delta)k}}{2^{(\beta-\delta)M}} \\
&\quad = \lim_{M\to\infty} \frac{\sum_{k=0}^{M+1} \omega(2^{-k}) 2^{(\beta-\delta)k} - \sum_{k=0}^{M} \omega(2^{-k}) 2^{(\beta-\delta)k}}{2^{(\beta-\delta)(M+1)} - 2^{(\beta-\delta)M}} \\
&\quad = \lim_{M\to\infty} \frac{\omega(2^{-M+1}) 2^{(\beta-\delta)(M+1)} - \omega(2^{-M+1}) 2^{(\beta-\delta)M}}{2^{(\beta-\delta)M}} = 0,
\end{aligned}$$

and recalling that $M > M_0$ for some large M_0, (3.3) follows for some modulus of continuity $\widetilde{\omega}$. Combining (3.2) and (3.3) we complete the proof of our statement. $\square$

We next prove that the function ψ^α is a supersolution near $\partial\Omega$.

Lemma 3.4. *Let $s \in (0, 1)$, $\alpha \in (0, s)$, and L be as in (1.2)–(1.3)–(1.4). Then for any C^1 domain Ω and for any function ψ satisfying (3.1), there is $\varrho_0 > 0$ such that*

$$-L(\psi^\alpha) \ge c_0 d^{\alpha-2s} > 0 \quad \text{in } \{0 < d < \varrho_0\}$$

for some constant $c_0 > 0$ depending only on N, s, α, Ω, ω, and the ellipticity constants.

Proof. Let $\alpha \in (0, s)$, $x_0 \in \Omega$, and $\varrho := d(x_0)$. We assume $\varrho \in (0, \varrho_0)$ for some $\varrho_0 > 0$ small which will be chosen later, and we consider the function

$$l(x) := \big(\psi(x_0) + \nabla\psi(x_0) \cdot (x - x_0)\big)_+,$$

satisfying $l(x_0) = \psi(x_0)$ and $\nabla\psi(x_0) = \nabla l(x_0)$. Notice that we can also assume $l > 0$ in $B_{\varrho/2}(x_0)$ and so we obtain

$$-L(l^\alpha)(x_0) = k l^{\alpha-2s}(x_0) = k\psi^{\alpha-2s}(x_0) \ge C\varrho^{\alpha-2s}, \tag{3.4}$$

where we have used the assumptions on ψ and set $k := l^{2s-\alpha}(x_0)[-L(l^\alpha)(x_0)] = -L(l^\alpha)(1) > 0$ (thanks to the homogeneity of l^α and [16, Lemma 2.3]).

Now, from Lemma 3.2 we know that

$$|\psi - l|(x_0 + y) \le \omega(|y|)|y|, \quad y \in \mathbb{R}^N,$$

and so, since $|a^\alpha - b^\alpha| \le C|a - b|(a^{\alpha-1} + b^{\alpha-1})$ for all $a, b \ge 0$, it follows that

$$|\psi^\alpha - l^\alpha|(x_0 + y) \le C(\psi^{\alpha-1} + l^{\alpha-1})(x_0 + y)\omega(|y|)|y|, \quad y \in \mathbb{R}^N, \tag{3.5}$$

for some $C > 0$ depending only on Ω and α, where we have used the first inequalities in (3.1). Furthermore, thanks to the properties of ψ, we have

$$|D^2(\psi^\alpha - l^\alpha)| \le C\omega(\varrho)\varrho^{\alpha-2} \quad \text{in } B_{\varrho/2}(x_0) \tag{3.6}$$

for some new $C > 0$, which implies

$$|\psi^\alpha - l^\alpha|(x_0 + y) \le \|D^2(\psi^\alpha - l^\alpha)\|_{L^\infty(B_{\varrho/2(x_0)})}|y|^2 \le C\omega(\varrho)\varrho^{\alpha-2}|y|^2 \tag{3.7}$$

for $y \in B_{\varrho/2}(x_0)$. To check the validity of (3.6), we compute

$$\begin{aligned}(\psi^\alpha - l^\alpha)_{x_j x_i}(x) = \alpha(\alpha-1)&\left[\psi^{\alpha-2}(x)\psi_{x_i}(x)\psi_{x_j}(x) - l^{\alpha-2}(x)\psi_{x_i}(x_0)\psi_{x_j}(x_0)\right]\\ &+ \alpha\psi^{\alpha-1}(x)\psi_{x_i x_j}(x),\end{aligned}$$

and we notice that $|\psi^{\alpha-1}\psi_{x_i x_j}| \le C\omega(\varrho)\varrho^{\alpha-2}$ from (3.1). On the other hand, we have

$$\begin{aligned}\psi^{\alpha-2}(x)\psi_{x_i}(x)\psi_{x_j}(x) - &l^{\alpha-2}(x)\psi_{x_i}(x_0)\psi_{x_j}(x_0)\\ &= \psi^{\alpha-2}(x)\left[\psi_{x_i}(x)\psi_{x_j}(x) - \psi_{x_i}(x_0)\psi_{x_j}(x_0)\right]\\ &\quad + \left[\psi^{\alpha-2}(x) - l^{\alpha-2}(x)\right]\psi_{x_i}(x_0)\psi_{x_j}(x_0),\end{aligned}$$

and so, since $\nabla\psi$ is continuous up to $\partial\Omega$ with modulus of continuity $\omega(\cdot)$, it follows that

$$\left|\psi^{\alpha-2}(x)\left[\psi_{x_i}(x)\psi_{x_j}(x) - \psi_{x_i}(x_0)\psi_{x_j}(x_0)\right]\right| \le C\omega(\varrho)\varrho^{\alpha-2}, \quad x \in B_{\varrho/2}(x_0).$$

Further, since $|D^2\psi(x)| \le C\omega(d(x))d^{-1}(x)$, we obtain

$$\begin{aligned}\left|\left[\psi^{\alpha-2}(x) - l^{\alpha-2}(x)\right]\psi_{x_i}(x_0)\psi_{x_j}(x_0)\right| &\le C|\psi(x) - l(x)|\left|\psi^{\alpha-3}(x) + l^{\alpha-3}(x)\right|\\ &\le C|D^2\psi(x_0)||x - x_0|^2\left|\psi^{\alpha-3}(x) + l^{\alpha-3}(x)\right|\\ &\le C\omega(\varrho)\varrho^{\alpha-2}, \quad x \in B_{\varrho/2}(x_0),\end{aligned}$$

and so (3.6) follows. Finally, since $\alpha \in (0, s)$ and $\psi = 0$ in $\mathbb{R}^N \setminus \Omega$,

$$|\psi^\alpha - l^\alpha|(x_0 + y) \le C|y|^s, \quad y \in \mathbb{R}^N \setminus B_1. \tag{3.8}$$

Consequently, if $\alpha + \gamma \neq 2s$, using (3.4), (3.5), (3.7), and (3.8) it follows that

$$\begin{aligned}-L(\psi^\alpha)(x_0) &= -L(l^\alpha)(x_0) - L(\psi^\alpha - l^\alpha)(x_0) \ge C\varrho^{\alpha-2s} - L(\psi^\alpha - l^\alpha)(x_0)\\ &= C\varrho^{\alpha-2s} - \int_{\mathbb{R}^N}(\psi^\alpha - l^\alpha)(x_0 + y)\frac{a(y/|y|)}{|y|^{N+2s}}dy\\ &\ge C\varrho^{\alpha-2s} - C\int_{\mathbb{R}^N}|\psi^\alpha - l^\alpha|(x_0 + y)\frac{dy}{|y|^{N+2s}}\\ &\ge C\varrho^{\alpha-2s} - C\omega(\varrho)\varrho^{\alpha-2}\int_{B_{\varrho/2}}\frac{dy}{|y|^{N+2s-2}} - C\int_{\mathbb{R}^N\setminus B_1}\frac{dy}{|y|^{N+s}}\\ &\quad - C\int_{B_1\setminus B_{\varrho/2}}(d^{\alpha-1} + l^{\alpha-1})(x_0 + y)\frac{\omega(|y|)}{|y|^{N+2s-1}}dy\\ &\ge C\varrho^{\alpha-2s} - C\omega(\varrho)\varrho^{\alpha-2s} - C - C\left(1 + \widetilde{w}(\varrho)\varrho^{\alpha-2s}\right) \ge C\varrho^{\alpha-2s}\end{aligned}$$

for some new constant $C > 0$ and all $0 < \varrho < \varrho_0$, where $\varrho_0 > 0$ depends only on N, s, α, Ω, ω, and the ellipticity constants. Notice that we have applied Lemma 3.3 twice (once to $d(\cdot)$, once to $l(\cdot)$). □

Proof of Theorem 1.1. Dividing g and u by a constant if necessary, we may assume $C_0 = 1$. Thanks to Lemma 2.1, we have $\|u\|_{L^\infty(\Omega)} \le C$. On the other hand, let $\overline{g}$ denote the extension of g given by Lemma 2.2. Then, the function

$$v = u - \overline{g}$$

solves (1.7) with $f := L\overline{g}$. Moreover, thanks to Lemma 2.4 we have $|f| \le C d^{\alpha-2s}$ in Ω.

Step 1. We claim that

$$|v| \le C d^{\alpha} \quad \text{in} \quad \Omega. \tag{3.9}$$

To prove this, we consider the function

$$\varphi(x) := M\psi^{\alpha}(x), \quad x \in \mathbb{R}^N,$$

where ψ is given by Lemma 3.1. Thanks to Lemma 3.4, for some $\varrho_0 > 0$ we have

$$-L\varphi \ge MCd^{\alpha-2s} \quad \text{in } \Omega_{\varrho_0} := \left\{x \in \Omega : 0 < d(x) < \varrho_0\right\}$$

for some constant $C > 0$ depending only on N, s, α, Ω, and the ellipticity constants. We now compare φ with v:

- We clearly have $\varphi = v = 0$ in $\mathbb{R}^N \setminus \Omega$ for all $M > 0$.
- Since $\|v\|_{L^\infty(\Omega)} \le C$ and $\psi^\alpha > 0$ in Ω, taking $M > 0$ large enough we have
$$\varphi \ge v \quad \text{in } \Omega \setminus \Omega_{\varrho_0}.$$
- Thanks to Lemma 3.4, we may choose $M > 0$ large enough such that
$$-L\varphi \ge -Lv \quad \text{in } \Omega_{\varrho_0}.$$

Consequently, taking M large enough (depending only on n, s, α, Ω, λ, and Λ), it follows from the comparison principle that $v \le \varphi$ in Ω. Repeating the above argument with $-v$, (3.9) follows.

Step 2. We next claim that

$$[v]_{C^{0,\alpha}(B_r(x_0))} \le C \tag{3.10}$$

for any ball $B_r(x_0) \subset \Omega$ with $d(x_0) = 2r$, and some constant C independent of $x_0 \in \Omega$ and $r > 0$.

To do this, we recall[2] that if w is a solution to $-Lw = f$ in B_2, then

$$[w]_{C^{0,\alpha}(B_1)} \le C\left(\|f\|_{L^\infty(B_2)} + \int_{\mathbb{R}^N} \frac{|w(x)|}{1+|x|^{N+2s}}\,dx\right). \tag{3.11}$$

Now, for any $x_0 \in \Omega$ and $r := d(x_0)/2$, we take $w(x) := v(x_0 + rx)$, which satisfies

$$-Lw(x) = f_r(x) := r^{2s} f(x_0 + rx) \le Cr^{2s} d^{\alpha-2s}(x_0 + rx) \le Cr^{\alpha} \quad \text{in } B_2,$$

since $x_0 + rx \in B_{2r}(x_0)$. On the other hand, since $|v| \le Cd^\alpha$ in Ω (thanks to Step 1), we have

$$|w(x)| = |v(x_0 + rx)| \le Cd^{\alpha}(x_0 + rx) \le Cr^{\alpha}(1 + |x|^{\alpha}) \quad \text{in } \mathbb{R}^N,$$

where $C > 0$ is a new constant independent of $x_0 \in \Omega$ and $r > 0$, and so

$$\int_{\mathbb{R}^N} \frac{|w(x)|}{1+|x|^{N+2s}}\,dx \le Cr^{\alpha} \int_{\mathbb{R}^N} \frac{1+|x|^{\alpha}}{1+|x|^{N+2s}}\,dx \le Cr^{\alpha},$$

[2] This follows for example from [17, Theorem 1.1] and [7, Theorem 5.1].

since $\alpha < 2s$. Above, we have used the fact that

$$d(x_0+rx) = \inf_{y\in\partial\Omega} |y-(x_0+rx)| \le \inf_{y\in\partial\Omega} |y-x_0| + r|x| = d(x_0)+r|x| = r(2+|x|),$$

whenever $x_0+rx\in\Omega$. Consequently, applying (3.11), we obtain

$$[w]_{C^{0,\alpha}(B_1)} \le Cr^\alpha,$$

from which (3.10) immediately follows.

Step 3. We can now finish the proof. Indeed, take $x, y\in\overline{\Omega}$, with $r=|x-y|$ and $\varrho = \min\{d(x), d(y)\}$. There are two possibilities:

- If $\varrho \le 2r$, assuming for instance $\varrho = d(y)$ and recalling (3.9), we have

$$|v(x)-v(y)| \le |v(x)|+|v(y)| \le C\,[d^\alpha(x)+d^\alpha(y)] \le C\,[(\varrho+r)^\alpha + \varrho^\alpha] \le \widetilde{C}r^\alpha$$

  for some constant $\widetilde{C}>0$ independent of $x,y\in\Omega$ and $r>0$.
- If $\varrho > 2r$ and $\varrho = d(y)$, it follows that $B_{2r}(x)\subset\Omega$, and thus thanks to (3.10)

$$|v(x)-v(y)| = r^\alpha\frac{|v(x)-v(y)|}{|x-y|^\alpha} \le r^\alpha \sup_{y\in B_r(x)} \frac{|v(x)-v(y)|}{|x-y|^\alpha} \le r^\alpha [v]_{C^{0,\alpha}(B_r(x))} \le \overline{C}r^\alpha.$$

Putting together the last two inequalities we find

$$|v(x)-v(y)| \le C|x-y|^\alpha \quad \text{for all } x,y\in\overline{\Omega}.$$

This implies that $v\in C^{0,\alpha}(\overline{\Omega})$ and, therefore, that $u\in C^{0,\alpha}(\overline{\Omega})$. ☐

Proof of Proposition 1.2. Let $\Omega := \mathbb{R}^2_+ := \{(x_1,x_2)\in\mathbb{R}^2 : x_2>0\}$ and u solve

$$\begin{cases} (-\Delta)^s u &= 0 & \text{in } \Omega, \\ u &= g & \text{in } \mathbb{R}^2\setminus\Omega, \end{cases} \tag{3.12}$$

with $g(x) := \min\{|x|^s, 1\}$. The solution to (3.12) is given by the Poisson kernel

$$u(x_1,x_2) = c_s x_2^s \int_{\mathbb{R}}\int_{-\infty}^0 \frac{g(z_1,z_2)}{|z_2|^s[(x_1-z_1)^2+(x_2-z_2)^2]} dz_1dz_2;$$

see [2]. Notice that, since $g\ge 0$, we clearly have

$$u(x_1,x_2) \ge c_s x_2^s \int_S \frac{g(z_1,z_2)}{|z_2|^s[z_1^2+(t-z_2)^2]} dz_1dz_2,$$

where $S := \{x_2 \le -|x_1|\}\cap B_1$. Thus, in order to prove that $u\notin C^{0,s}(\overline{\Omega})$, it is enough to show that the last integral is unbounded for $x_1=0$ and $x_2>0$ small.

For this, we set $t=x_2$, and we consider

$$\begin{aligned}\int_S \frac{g(z_1,z_2)}{|z_2|^s[z_1^2+(t-z_2)^2]} dz_1dz_2 &= \int_S \frac{(z_1^2+z_2^2)^{\frac{s}{2}}}{|z_2|^s[z_1^2+(t-z_2)^2]} dz_1dz_2 \\ &= \int_0^1\int_{\frac{5}{4}\pi}^{\frac{7}{4}\pi} \frac{r^{s+1}}{r^s|\sin\theta|^s(r^2+2r|\sin\theta|t+t^2)} d\theta dr \\ &= \int_{\frac{5}{4}\pi}^{\frac{7}{4}\pi} |\sin\theta|^{-s}\int_0^1 \frac{r}{r^2+2r|\sin\theta|t+t^2} drd\theta,\end{aligned}$$

where we have passed to polar coordinates $z_1 = r\cos\theta$, $z_2 = r\sin\theta$. Since $|\sin\theta| \in [\sqrt{2}/2, 1]$, it follows that

$$\int_0^1 \frac{r}{r^2 + 2r|\sin\theta|t + t^2}dr \geq \int_0^1 \frac{r}{r^2 + 2rt + t^2}dr = \frac{1}{2}\int_0^1 \frac{2(r+t)}{(r+t)^2}dr - t\int_0^1 \frac{dr}{(r+t)^2}$$
$$= \ln\left(\frac{1}{t}\right) + \ln(1+t) - \frac{1}{1+t} = \ln\left(\frac{1}{t}\right) + O(1)$$

as $t \to 0^+$. Consequently, setting $\kappa_s := \int_{\frac{5}{4}\pi}^{\frac{7}{4}\pi} |\sin\theta|^{-s}$, we obtain

$$u(0,t) \geq c_s t^s \int_S \frac{g(z_1,z_2)}{|z_2|^s[z_1^2 + (t-z_2)^2]}dz_1 dz_2 \geq c_s \kappa_s t^s \ln\left(\frac{1}{t}\right) + O(t^s), \quad \text{as } t \to 0^+,$$

which shows that $u \notin C^{0,s}(\overline{\Omega})$.

Finally, notice that using such function u one can actually construct a counterexample in a bounded domain, too. $\square$

Proof of Proposition 1.3. As in the proof of Theorem 1.1, we have that $v = u - \overline{g}$ solves (1.7), with $|f| \leq Cd^{\alpha-2s}$ in Ω. Setting $\theta := \min\{\alpha - s, \gamma\} > 0$, we can apply [18, Theorem 1.2, Proposition 3.2] (cf. [18, Remark 3.4]) to conclude that $v \in C^{0,s}(\overline{\Omega})$. $\square$

4. Lipschitz domains

We focus now on the case in which Ω is Lipschitz. Again, the idea is to construct a suitable supersolution for problem (1.7) exploiting that the r.h.s. explodes as $d^{\beta-2s}$ near the boundary $\partial\Omega$ for some $\beta \in (0,1)$. Since the domain is only Lipschitz and the operator L is not homogeneous, the strategy followed in the above section cannot work in this more general framework. The construction of a new barrier is the main difficulty here, and it works as follows.

For any fixed direction $e \in \mathbb{S}^{N-1}$ and $\eta > 0$, we consider the function

$$\psi(x) := e\cdot x + \eta|x|\left(1 - \frac{(e\cdot x)^2}{|x|^2}\right), \quad x \in \mathbb{R}^N,$$

and, for any $\beta \in (0,1)$, we define

$$\Phi_\beta := (\psi_+)^\beta. \tag{4.1}$$

Notice that $\Phi_\beta \in C^{0,\beta}(\mathbb{R}^N)$ and it is positive in the cone

$$\mathcal{C}_{-\eta} := \left\{x \in \mathbb{R}^N : \frac{e\cdot x}{|x|} > -\eta\left(1 - \frac{(e\cdot x)^2}{|x|^2}\right)\right\}, \tag{4.2}$$

while zero otherwise. We begin with the following lemma (cf. also [8,19]).

Lemma 4.1. *Let $s \in (0,1)$, $e \in \mathbb{S}^{N-1}$, and L be as in (1.2)–(1.3). Then for all $\eta > 0$, there exists $\beta_0 \in (0,1)$ such that the function Φ_β defined in (4.1) satisfies*

$$\begin{cases} -L\Phi_\beta \geq c_0 d^{\beta-2s} > 0 & \text{in } \mathcal{C}_{-\eta}, \\ \Phi_\beta = 0 & \text{in } \mathbb{R}^N \setminus \mathcal{C}_{-\eta} \end{cases} \tag{4.3}$$

for all $\beta \in (0,\beta_0]$ and some $c_0 > 0$. The constants β_0 and c_0 depend only on n, s, λ, Λ, and η.

Proof. As in [18, Lemma 4.1], using the homogeneity of Φ_β and $d^{\beta-2s}$, and the properties of the kernel K, it is enough to prove

$$-L\Phi_\beta \geq C > 0 \quad \text{in } e + \partial\mathcal{C}_{-\eta}.$$

This is because $\{\lambda(e+\partial\mathcal{C}_{-\eta})\}_{\lambda>0} = \mathcal{C}_{-\eta}$.

Now, let us take $\varrho = \varrho(\eta) > 0$ so that

$$0 < \varrho < \inf_{\substack{x\in\partial\mathcal{C}_{-\eta}\\ y\in e+\partial\mathcal{C}_{-\eta}}} |x-y| := \operatorname{dist}\left(\partial\mathcal{C}_{-\eta}, e+\partial\mathcal{C}_{-\eta}\right),$$

and notice that

$$|\nabla\psi| \leq C \quad \text{and} \quad |D^2\psi| \leq C|x|^{-1} \quad \text{in } \mathbb{R}^N\setminus\{0\}$$

for some constant C depending on η. Consequently, it follows that

$$\|\Phi_\beta\|_{C^2(B_\varrho(x))} \leq C \quad \text{for all } x\in e+\partial\mathcal{C}_{-\eta}, \tag{4.4}$$

for some new constant $C>0$ independent of x. On the other hand, we have

$$\Phi_\beta \to \Phi_0 := \chi_{\mathcal{C}_{-\eta}} \quad \text{pointwise in } \mathbb{R}^N \tag{4.5}$$

as $\beta\to 0^+$, and moreover

$$|\Phi_\beta(x) - \Phi_\beta(x-y)| \leq C(1+|y|^{\beta_0}) \quad \text{for all } x\in e+\mathbb{R}^N\setminus\mathcal{C}_{-\eta},\ y\in\mathbb{R}^N, \tag{4.6}$$

for all $\beta\in(0,\beta_0]$ and some constant C independent of $x,y\in\mathbb{R}^N$ and independent of β. Finally, let us define

$$h_\beta(x,y) := \left(\Phi_\beta(x) - \frac{\Phi_\beta(x+y)+\Phi_\beta(x-y)}{2}\right)K(y).$$

Thus, using (4.4) and (4.6), we obtain

$$|h_\beta(x,y)| \leq C|y|^{-N-2(1-s)} \quad \text{for all } x\in e+\partial\mathcal{C}_{-\eta},\ y\in B_\varrho,$$

and

$$|h_\beta(x,y)| \leq C\left(1+|y|^{-N-2s+\beta_0}\right) \quad \text{for all } x\in e+\partial\mathcal{C}_{-\eta},\ y\in\mathbb{R}^N,$$

for some constant C independent of x and y (actually it depends only on n, s, λ, Λ, α, and Ω). Recalling (4.5), we can apply the Lebesgue dominated convergence theorem to deduce

$$-L\Phi_\beta(x) = \int_{B_\varrho} h_\beta(x,y)dy + \int_{\mathbb{R}^N\setminus B_\varrho} h_\beta(x,y)dy \to -L\Phi_0(x), \tag{4.7}$$

as $\beta\to 0$, for every fixed $x\in e+\partial\mathcal{C}_{-\eta}$. Actually, the above convergence is uniform w.r.t. $x\in e+\partial\mathcal{C}_{-\eta}$ and L. Indeed, assume by contradiction that for any sequence $\beta_j\to 0^+$, there exist a sequence $\{x_j\}_j\in e+\partial\mathcal{C}_{-\eta}$ and a sequence of operators $\{L_j\}_j$ satisfying (1.2)–(1.3) such that

$$|L_j\Phi_{\beta_j}(x_j) - L_j\Phi_0(x_j)| \geq \varepsilon \tag{4.8}$$

for some $\varepsilon>0$ and all $j\in\mathbb{N}$. Then, using again the bounds in (4.4) and (4.6), we easily deduce that $|L_j\Phi_{\beta_j}(x_j)|\leq C$ for some constant C independent of $j\in\mathbb{N}$ and so, up to passing to a subsequence, we obtain that $L_j\Phi_{\beta_j}(x_j)$ has a finite limit as $j\to+\infty$. Further, using the pointwise convergence in (4.5) and a standard diagonal procedure, we can extract a subsequence $\{\beta_{j_k}\}_k\subset\{\beta_j\}_j$ for which $|L_k\Phi_{\beta_{j_k}}(x_k) - L_k\Phi_0(x_k)|\leq\varepsilon/2$ for all $k\in\mathbb{N}$, obtaining a contradiction with (4.8).

On the other hand, since $\Phi_0 = \chi_{\mathcal{C}_{-\eta}}$, we obtain

$$-L\Phi_0(x) = \int_{\mathbb{R}^N} (\Phi_0(x) - \Phi_0(y)) K(x-y)dy = \int_{\mathbb{R}^N \setminus \mathcal{C}_{-\eta}} K(x-y)dy.$$

Consequently, writing $x = e + P$ with $P \in \partial\mathcal{C}_{-\eta}$ and noticing that $\mathbb{R}^N \setminus \mathcal{C}_{-\eta} \subset -P + \mathbb{R}^N \setminus \mathcal{C}_{-\eta}$, it follows that

$$-L\Phi_0(x) = \int_{-P+\mathbb{R}^N \setminus \mathcal{C}_{-\eta}} K(y-e)dy \geq \int_{\mathbb{R}^N \setminus \mathcal{C}_{-\eta}} K(y-e)dy \geq c > 0$$

for some $c > 0$ independent of $x \in e + \partial\mathcal{C}_{-\eta}$. Thus, recalling the uniform convergence in (4.7), we deduce the existence of a small $\beta_0 \in (0,1)$ such that

$$-L\Phi_\beta(x) \geq c/2 > 0$$

for all $x \in e + \partial\mathcal{C}_{-\eta}$ and $\beta \in (0, \beta_0]$. ☐

Proof of Theorem 1.4. Let $\beta_0 > 0$ be given by Lemma 4.1, and assume without loss of generality that $\beta_0 \leq \alpha_0$, where α_0 is given by Lemma 2.3.

Let $v = u - \overline{g}$, where $\overline{g}$ is chosen as in Lemma 2.3. Thanks to Lemma 2.4(b), we have $|L\overline{g}| \leq Cd^{\alpha-2s}$ in Ω, and so

$$|Lv| \leq Cd^{\alpha-2s} \quad \text{in } \Omega, \tag{4.9}$$

since v satisfies (1.7) with $f = L\overline{g}$. We want to prove that

$$|v(x)| \leq C|x - z_0|^\alpha = Cd^\alpha(x) \quad \text{in } \Omega, \tag{4.10}$$

where $z_0 \in \partial\Omega$ is the projection of x on $\partial\Omega$. To do this, since $\partial\Omega$ is Lipschitz we have that for any point $z_0 \in \partial\Omega$, there are $r, \eta > 0$ such that

$$B_{2r}(z_0) \cap \Omega \subseteq B_{2r}(z_0) \cap (z_0 + \mathcal{C}_{-\eta}),$$

where $\mathcal{C}_{-\eta}$ is defined in (4.2). Moreover, we can choose $r > 0$ and $\eta > 0$ independently of $z_0 \in \Omega$ (i.e., $x \in \Omega$).

Now, we consider the truncation

$$w := v\chi_{B_{2r}(z_0)},$$

which satisfies $|Lw| \leq Cd^{\alpha-2s}$ in $B_r(z_0) \cap \Omega$, thanks to (4.9).

On the other hand, we consider the function

$$\varphi(x) := M\Phi_\alpha(x - z_0), \quad x \in \mathbb{R}^N,$$

where Φ_α is defined in (4.1), and $M > 0$ is to be chosen. Thanks to Lemma 4.1,

$$-L\varphi \geq Mc_0 d^{\alpha-2s} > 0 \quad \text{in } B_r(z_0) \cap (z_0 + \mathcal{C}_{-\eta}).$$

Now, choose $M > 0$ so that

$$\varphi \geq w \quad \text{in } \mathbb{R}^N \setminus (B_r(z_0) \cap \Omega),$$

which is possible since w is bounded, and so that

$$-L\varphi \geq Mc_0 d^{\alpha-2s} \geq Cd^{\alpha-2s} \geq -Lw \quad \text{in } B_r(z_0) \cap \Omega.$$

By the maximum principle, we deduce

$$v = w \leq C\varphi \leq C|x - z_0|^\beta = Cd^\beta(x) \quad \text{in } B_r(z_0) \cap \Omega$$

and, repeating the above argument with w replaced by $-w$, (4.10) follows.

To finish the proof, we can repeat the proof of Theorem 1.1, combining (4.10) with the interior estimate (3.11). ☐

As a consequence, we find the following corollary.

Corollary 4.2. *Let N, Ω, s, L, and g be as in Theorem* 1.4. *Then, there exists a viscosity solution $u \in C(\overline{\Omega})$ to* (1.1).

Proof. Let $\Omega_\varepsilon \subset\subset \Omega$ be a sequence of smooth domains such that $\Omega_\varepsilon \to \Omega$ in the Hausdorff distance, and such that Ω_ε are Lipschitz sets (uniformly in ε).

Let $\bar{g}$ be a Hölder continuous extension of g inside Ω, and let $g_\varepsilon \in C_c^\infty(\mathbb{R}^N)$ be such that $g_\varepsilon \to \bar{g}$ uniformly in Ω, $g_\varepsilon \to g$ a.e. in $\mathbb{R}^N \setminus \Omega$, and such that (1.5) holds uniformly in ε.

Then, by [14, Theorem 5.6] there exists a viscosity solution u_ε to $Lu_\varepsilon = 0$ in Ω_ε, $u_\varepsilon = g_\varepsilon$ in Ω_ε^c. By the Theorem 1.4, we have a uniform bound $\|u_\varepsilon\|_{C^{0,\alpha}(\overline{\Omega})} \le C$, with $\alpha > 0$. By the Arzelà-Ascoli theorem, we have $u_\varepsilon \to u$ uniformly in $\overline{\Omega}$, up to subsequence, where $u \in C^{0,\alpha}(\overline{\Omega})$. Moreover, $u_\varepsilon \to u$ almost everywhere in $\mathbb{R}^N \setminus \Omega$, where $u := g$ outside Ω. Then, by stability of viscosity solutions (see, e.g., [6, Lemma 4.5]), we have that u is a viscosity solution of (1.1). $\square$

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